International Journal of Theoretical Physics, Vol. 30, No. 2, 1991

Generalized Noether Theorems and Applications

Li Zi-Ping[1] **and Li Xin**[2]

Received March 17, 1990

We generalize the first and second Noether theorems (Noether identities) to a constrained system in phase space. As an example, the conservation law deriving from Lagrange's formalism cannot be obtained from H_E via the generalized first Noether theorem (GFNT); Dirac's conjecture regarding secondary first-class constraints (SFCC) is invalid in this example. A preliminary application of the generalized Noether identities (GNI) to nonrelativistic charged particles in an electromagnetic field shows that on the constrained hypersurface in phase space one obtains electric charge conservation. This conservation law is valid whether Dirac's conjecture holds true or not.

1. INTRODUCTION

The connection between the invariance of the action integral under a finite continuous group and conservation laws is given by the first Noether theorem (FNT). The second Noether theorem refers to the invariance of the action under an infinite continuous group. In this case there exist differential identities which involve variational derivatives. These theorems have an important role in theoretical physics. A generalization of the FNT was given by Rosen (1974*a*,*b* and references therein), and a generalization of the FNT for constrained and nonconservative systems was given by Li (1981, 1984; Li and Li, 1990). A generalization of Noether's identities for variant systems was given by Li (1987). In these papers, all considerations are based on an examination of the Lagrangian in configuration space and the corresponding transformation expressed in terms of Lagrange variables. For regular Lagrangians of classical mechanical systems the invariance of the Lagrangian under a finite continuous group in terms of Hamilton's variables was discussed by Djukic (1974). Here, we further discuss singular Lagrangian systems. Dirac (1950, 1964) proposed a method for developing

[1]Department of Applied Physics, Beijing Polytechnic University, Beijing, 100022, China.
[2]Space Physics Institute, Chinese Academy of Science, Beijing, China.

0020-7748/91/0200-0225$06.50/0

the formalism for this system, and its quantization. The singular Lagrangian system is subject to some inherent phase space constraint. Dirac conjectured that all secondary first-class constraints (SFCC) generate gauge transformations which leave the physical state invariant. Dirac's conjecture has been widely discussed. In this paper, first, we generalize the FNT in phase space for a constrained Hamiltonian system. An example is given in which the conservation law deriving from the usual Lagrange formalism cannot be obtained from H_E via this GFNT, which implies that Dirac's conjecture fails in this example. Second, considering the transformation properties of the system under an infinite continuous group in terms of canonical variables, we obtain the GNI in phase space. Combining these GNI and constraint conditions, we obtain more relations among some of the variables. A preliminary application of the GNI to nonrelativistic charged particles in an electromagnetic field, on the constrained hypersurface, shows that one obtains electric charge conservation automatically, which differs from the usual way to obtain this result. This conservation law is valid whether Dirac's conjecture holds or not.

2. GENERALIZATION OF FIRST NOETHER THEOREM

Consider the transformation properties of a constrained dynamic system under a finite continuous group. We can generalize the FNT to Hamiltonian coordinates. For simplicity one usually considers a system with finite degrees of freedom exhibiting the essential problems of invariant theories; the extension to field theories is formally straightforward. Consider a mechanical system with N degrees of freedom described by a singular Lagrangian $L(t, q, q)$ $(q=\{q^1, \ldots, q^N\})$. This system is subject to Dirac's constraints

$$G_k(q, p)=0 \qquad (k=1,2,\ldots,K) \tag{1}$$

where $p=\{p_1, \ldots, p_N\}$ are the generalized momenta corresponding to the generalized coordinates q. Let us consider an infinitesimal continuous r-parameter transformation of the time, generalized coordinates, and generalized momenta

$$\begin{aligned} t &\to t' = t+\delta t = t+\varepsilon_\sigma \tau^\sigma(t, q, p) \\ q^i(t) &\to q^{i'}(t') = q^i(t)+\delta q^i(t) = q^i(t)+\varepsilon_\sigma \xi^{i\sigma}(t, q, p) \\ p_i(t) &\to p_i'(t') = p_i(t)+\delta p_i(t) = p_i(t)+\varepsilon_\sigma \eta_i^\sigma(t, q, p) \end{aligned} \tag{2}$$

Suppose $L_{PS}=p_i\dot{q}^i-H$ is gauge variant under the transformation (2), i.e., is invariant up to an exact differential $\varepsilon_\sigma\, d\Omega^\sigma/dt$, where H is the Hamiltonian, $\Omega^\sigma=\Omega^\sigma(t, q, p)$ $(\sigma=1,2,\ldots,r)$, and ε_σ are parameters. Repeated indices are summed. Then we have

$$\bar{\delta}p_i\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) + \bar{\delta}q^i\left(-\dot{p}_i - \frac{\partial H}{\partial q^i}\right) + \frac{d}{dt}(p_i\bar{\delta}q^i + L_{\text{PS}}\delta t) = \varepsilon_\sigma \frac{d\Omega^\sigma}{dt} \tag{3}$$

where

$$\bar{\delta}p_i = \delta p_i - \dot{p}_i\delta t, \qquad \bar{\delta}q^i = \delta q^i - \dot{q}^i\delta t \tag{4}$$

Under the transformation (2), suppose the changes of the G_k are given by $\delta G_k = \varepsilon_\sigma K_K^\sigma$. Then we have

$$\bar{\delta}G_k = \frac{\partial G_k}{\partial q^i}\bar{\delta}q^i + \frac{\partial G_k}{\partial p_i}\bar{\delta}p_i = \varepsilon_\sigma F_k^\sigma \tag{5}$$

where

$$F_k^\sigma = K_k^\sigma - \frac{\partial G_k}{\partial q^i}\dot{q}^i\tau^\sigma - \frac{\partial G_k}{\partial p_i}\dot{p}_i\tau^\sigma \tag{6}$$

Using a Lagrangian multiplier $\lambda^k(t)$ and combining the expressions (3) and (5), from the equations of motion of a constrained Hamiltonian system, one obtains

$$\frac{d}{dt}[p_i(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) + L_{\text{PS}}\tau^\sigma - \Omega^\sigma] = \lambda^k F_k^\sigma \tag{7}$$

Therefore, we have the following GFNT in Hamilton form: If, under the transformation (2), the L_{PS} is invariant up to an exact differential and such that constraint conditions satisfy $F_k^\sigma = 0$, then automatically there are conservation laws

$$p_i\xi^{i\sigma} - H\tau^\sigma - \Omega^\sigma = \text{const} \tag{8}$$

This result is based upon the symmetry of the system in phase space.

3. DIRAC'S CONJECTURE

Dirac's generalized canonical formalism plays an important role in modern field theory (Sundermeyer, 1982). However, there are some basic problems in this theory that are still widely discussed in the literature. One of them is Dirac's conjecture. Using the above GFNT in phase space, we can give a counterexample to deny the validity of Dirac's conjecture.

Let us recall briefly the results of Dirac. For the singular Lagrangian L the Hessian matrix $H_{ij} = \partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$ has a rank M less than the number of degrees of freedom N. It is not possible to eliminate all the velocities $\dot{q}^i$ as functions of momenta p_i by using just their definition $p_i = \partial L/\partial\dot{q}^i$; this situation implies the existence of $N-M$ independent relations between the q's and p's of the form

$$\varphi_k(q, p) \approx 0 \tag{9}$$

These relations are called primary constraints (PC). To account automatically for these constraints we introduce the total Hamiltonian

$$H_T = H + \lambda^k \varphi_k \tag{10}$$

where λ^k is a set of Lagrangian multipliers. All the constraints must be preserved in time,

$$\dot{\varphi}_k = \{\varphi_k, H_T\} \approx 0 \tag{11}$$

where $\{\cdot, \cdot\}$ denote the Poisson bracket. This consistency requirement often implies new constraints

$$\chi_l(q, p) \approx 0 \tag{12}$$

etc. The process of the consistency requirements will terminate at some stage when new constraints no longer appear. Constraints thus obtained and different from PC are called secondary constraints (SC). The constraints in the first class are those whose Poisson bracket with any of the constraints is zero or equal to the linear combination of the constraints; if this is not the case, the constraint is called second class. Dirac conjectured that all SFCC are independent generators of gauge transformations which generate equivalence transformations among physical states. If this conjecture holds true, then the dynamics of a system possessing PC $\varphi_k \approx 0$ and SFCC $\chi_m \approx 0$ should be correctly described by the equations of motion arising from the extended Hamiltonian

$$H_E = H + \lambda^k \varphi_k + \mu^m \chi_m \tag{13}$$

where μ^m are also Lagrangian multipliers. Over a long period of time there have been objections to Dirac's conjecture (Dominici and Gomis, 1980; Castellani, 1982; Costa *et al.*, 1985; Sugano, 1982; Sugano and Kamo, 1982; Sugano and Kimura, 1983*a,b*). Due to an improper interpretation of the relation of gauge degrees of freedom to SFCC, there still appear arguments asserting that all SFCC are associated with gauge freedom (Gotoy, 1983; Di Stefano, 1983; Appleby, 1982). Grácia and Pons (1988) point out a careful distinction between the gauge transformation of a solution of the equation of motion and a gauge transformation of a point in phase space, which allows us to give a definitive clarification of the Dirac conjecture. Numerous examples are now known for which Dirac's conjecture fails (Allcock, 1975; Cawley, 1979, 1980; Frenkel, 1980). All these objections are based on the straightforward observation that the equations of motion deriving from H_E are not strictly equivalent to the corresponding Lagrange equations. Here, we shall observe conservation laws deriving from H_E via the GFNT in phase space which in some examples are exactly equivalent to the conservation laws arising from Lagrange's formalism via the classical FNT, and in others are not.

Now, from the Lagrangian

$$L = \dot{x}(\dot{z}_1^2 + \dot{z}_2^2) + \tfrac{1}{2}y[(z_1 - a)^2 + (z_2 - b)^2] \tag{14}$$

where a and b are constants. There is a PC

$$\varphi = p_y \approx 0 \tag{15}$$

and the total Hamiltonian has an arbitrary function λ,

$$H_T = [p_x(p_{z_1}^2 + p_{z_2}^2)]^{1/2} - \tfrac{1}{2}y[(z_1 - a)^2 + (z_2 - b)^2] + \lambda\varphi \tag{16}$$

The consistency condition (11) to equation (15) yields the additional SC,

$$\chi_1 = (z_1 - a)^2 + (z_2 - b)^2 \approx 0 \tag{17}$$

We do not write the constraint in linearized form, as Cawley and others do, because $\chi \approx 0$ implies $\chi^2 \approx 0$, which will confuse the concepts of weak and strong equality.

The consistency condition to equation (17) gives another SC

$$\chi_2 = (z_1 - a)p_{z_1} + (z_2 - b)p_{z_2} \approx 0 \tag{18}$$

The algorithm terminates after the next step, the consistency of equation (18), which gives

$$\chi_3 = p_x \approx 0 \tag{19}$$

The constraints φ, χ_1, χ_2, and χ_3 are first class. So the extended Hamiltonian is

$$H_E = H + \lambda\varphi + \mu_1\chi_1 + \mu_2\chi_2 + \mu_3\chi_3 \tag{20}$$

The Lagrangian L_{PS} and φ are invariant under rotation in the (z_1, z_2) plane around the point (a, b). From the GFNT expression (8) one can obtain angular momentum conservation. This conclusion agrees with the result formally yielded by Lagrange's variable via the classical FNT. But if we take into account the SFCC for this problem, from the extended Hamiltonian H_E we cannot obtain the above results. The Dirac conjecture fails for this example.

4. GENERALIZED NOETHER IDENTITIES IN PHASE SPACE

We shall give a generalization of Noether's identities in Hamilton form. Let us consider the invariance of the action integral under a transformation which depends upon the arbitrary functions $\varepsilon_\sigma(t)$ and their derivatives up to some fixed order. Let such an infinitesimal transformation in phase space be

$$\begin{aligned} t \to t' &= t + R^\sigma \varepsilon_\sigma(t) = t + a_k^\sigma(t) D^k \varepsilon_\sigma(t) \\ q^i(t) \to q^{i'}(t') &= q^i(t) + S^{i\sigma}\varepsilon_\sigma(t) = q^i(t) + b_l^{i\sigma} D^l \varepsilon_\sigma(t) \\ p_i(t) \to p_i'(t') &= p_i(t) + T_i^\sigma \varepsilon_\sigma(t) = p_i(t) + c_{im}^\sigma(t) D^m \varepsilon_\sigma(t) \end{aligned} \tag{21}$$

where $D^k = d^k/dt^k$, and $a_k^\sigma(t)$, $b_l^{i\sigma}(t)$, $c_{im}^\sigma(t)$ are some smoothed functions. Under the transformation (21), suppose the action integral with Lagrangian L_{PS} is gauge variant. Then we have

$$\int_{t_1}^{t_2} [E_{(p)}^i (T_i^\sigma - \dot{p}_i R^\sigma)\varepsilon_\sigma + E_i^{(q)}(S^{i\sigma} - \dot{q}^i R^\sigma)\varepsilon_\sigma]\, dt + [p_i(S^{i\sigma} - \dot{q}^i R^\sigma) + L_{\text{PS}} R^\sigma - \Omega^\sigma]\varepsilon_\sigma \Big|_{t_1}^{t_2} = 0 \tag{22}$$

where

$$E_{(p)}^i = \dot{q}^i - \frac{\partial H}{\partial p_i}, \qquad E_i^{(q)} = -\dot{p}_i - \frac{\partial H}{\partial q^i} \tag{23}$$

Ω^σ are linear differential operators. Since $\varepsilon_\sigma(t)$ are arbitrary, we may choose $\varepsilon_\sigma(t)$ and their derivatives up to some fixed order such that the endpoints vanish, and repeat the integration by part of the left-hand side of the identity (22). Again appealing to the arbitrariness of the $\varepsilon_\sigma(t)$, we can force the endpoint term to vanish. After this we can apply the fundamental lemma of the calculus of variations to conclude that

$$\tilde{T}_i^\sigma E_{(p)}^i - \tilde{R}^\sigma(E_{(p)}^i \dot{p}_i) + \tilde{S}^{i\sigma} E_i^{(q)} - \tilde{R}^\sigma(E_i^{(q)} \dot{q}^i) = 0 \tag{24}$$

where $\tilde{R}^\sigma$, $\tilde{S}^{i\sigma}$, $\tilde{T}_i^\sigma$ are the adjoint operators with respect to R^σ, $S^{i\sigma}$, T_i^σ, respectively (Li, 1987). The expressions (24) are called the GNI in phase space. As is well known, a gauge-variant system is a constrained dynamical system (Sundermeyer, 1982). We combine the constraint equations and the GNI (24), which may give rise to more relationships among some of the variables. Sometimes these can tell us at what stage the Dirac–Bergmann algorithm will terminate.

5. GAUGE VARIANCE AND ELECTRIC CHARGE CONSERVATION

For a system of nonrelativistic particles each having charge Q_i, mass m_i, and a displacement $\mathbf{r}_i(t)$ at time t in an electric field $\mathbf{E}$ and magnetic field **B**, the Lagrangian is (Kobe, 1981)

$$L=\frac{1}{2}\int d^3x\,(\mathbf{E}^2-\mathbf{B}^2)+\frac{1}{2}m_i\dot{\mathbf{r}}^2+\frac{1}{c}\int d^3x\,(\mathbf{J}\cdot\mathbf{A}-c\rho A_0) \tag{25}$$

Under the gauge transformation

$$\mathbf{A}'=\mathbf{A}+\nabla\varepsilon,\qquad A_0'=A_0-\frac{1}{c}\frac{\partial\varepsilon}{\partial t} \tag{26}$$

where $\varepsilon(\mathbf{r},t)$ is an arbitrary differentiable function, the Lagrangian is gauge-variant, i.e.,

$$L'=L+\sum_i\frac{Q_i}{c}\frac{d\varepsilon(\mathbf{r}_i,t)}{dt} \tag{27}$$

The canonical momenta conjugate to the coordinates $\mathbf{r}_i$ are

$$\mathbf{p}_i=\partial L/\partial\dot{\mathbf{r}}=m\dot{\mathbf{r}}_i+\frac{Q_i}{c}\mathbf{A} \tag{28}$$

The canonical momenta conjugate to the field are

$$\boldsymbol{\pi}=\delta L/\delta\dot{\mathbf{A}}=-\mathbf{E}/c,\qquad \pi_0=\delta L/\delta\dot{A}_0=0 \tag{29}$$

The Hamiltonian for the total system is

$$\begin{aligned}H&=\int d^3x\,(\boldsymbol{\pi}\cdot\dot{\mathbf{A}}+\pi_0\dot{A}_0)+\mathbf{p}_i\cdot\dot{\mathbf{r}}_i-L\\&=\frac{1}{2}\int d^3x\,(\mathbf{E}^2+\mathbf{B}^2)+\frac{1}{2m_i}\left(\mathbf{p}_i-\frac{Q_i}{c}\mathbf{A}\right)^2+\int d^3x\,(\rho-\nabla\cdot\mathbf{E})A_0\end{aligned} \tag{30}$$

The PC of this system is

$$\varphi=\pi_0\approx 0 \tag{31}$$

The consistency condition of φ provides the SC

$$\chi=\dot{\pi}_0=\{\pi_0,H_T\}=\nabla\cdot\mathbf{E}-\rho\approx 0 \tag{32}$$

where $H_T=H+\int d^3x\,\lambda\pi_0$, λ is a Lagrangian multiplier, and $\{\cdot,\cdot\}$ denotes the Poisson bracket,

$$\{F,G\}=\int d^3x\left[\frac{\delta F}{\delta A}\frac{\delta G}{\delta\pi}-\frac{\delta F}{\delta\pi}\frac{\delta G}{\delta A}\right] \tag{33}$$

In the phase space, the Lagrangian L_{PS} is also gauge-variant under the transformation

$$\delta t = 0, \qquad \delta \mathbf{r}_i = 0, \qquad \delta \mathbf{p}_i = \mathbf{T}_i \varepsilon = \frac{Q_i}{c} \nabla \varepsilon$$
$$\delta \mathbf{A} = \mathbf{S}\varepsilon = \nabla \varepsilon, \qquad \delta A_0 = S_0 \varepsilon = -\frac{1}{c}\frac{\partial \varepsilon}{\partial t} \tag{34}$$
$$\delta \boldsymbol{\pi} = 0, \qquad \delta \pi_0 = 0$$

In this case, the GNI (24) becomes

$$\mathbf{T}_i\left(\dot{\mathbf{r}}_i - \frac{\partial H}{\partial \mathbf{p}_i}\right) + \tilde{\mathbf{S}}\left(-\dot{\boldsymbol{\pi}} - \frac{\delta H}{\delta \mathbf{A}}\right) + \tilde{S}_0\left(-\dot{\pi}_0 - \frac{\delta H}{\delta A_0}\right) = 0 \tag{35}$$

Substituting the expressions (28), (30), and (34) into the identity (35), after taking the integral over the whole coordinate space, we conclude that on the constrained hypersurface the electric charge must be conserved. In the above discussion we have not used the canonical equations of the constrained dynamical system; hence this conservation law is valid whether Dirac's conjecture holds true or not.

6. CONCLUSIONS

We have obtained the GFNT and GNI for constrained dynamical systems in Hamilton variables, which may be used to analyze the Dirac constraints of the system. We observe that the conservation law deriving from H_E via the GFNT may or may not be exactly equivalent to the corresponding conservation law arising from Lagrange's formalism via the classical FNT. An example was given in which Dirac's conjecture fails, which differs from other counterexamples in that we do not write constraints in a linearized form as Cawley and others do. If we write the constraint in a linearized form, then all the constraints become second class. We combine the GNI for gauge-variant systems and constraint equations, which can give rise to more restrictions among some of the variables. As a preliminary application to nonrelativistic charged particles in an electromagnetic field, on the constrained hypersurface one obtains electric charge conservation automatically. This result does not depend on the validity of Dirac's conjecture.

REFERENCES

Allcock, G. R. (1975). *Philosophical Transactions of the Royal Society of London A*, **279**, 487–545.
Appleby, D. M. (1982). *Journal of Physics A*, **15**, 1191–1200.

Castellani, L. (1982). *Annals of Physics*, **143**, 357–371.
Cawley, R. (1979). *Physical Review Letters*, **42**, 413–416.
Cawley, R. (1980). *Physical Review D*, **21**, 2988–2990.
Costa, M. E. V., Girotti, H. O., and Simões, T. J. M. (1985). *Physical Review D*, **32**, 405–410.
Dirac, P. A. M. (1950). *Canadian Journal of Mathematics*, **2**, 129–148.
Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Yeshiva University Press, New York.
Di Stefano, R. (1983). *Physical Review D*, **27**, 1752–1765.
Djukic, D. S. (1974). *Archives of Mechanics*, **26**, 243–249.
Dominici, D., and Gomis, J. (1980). *Journal of Mathematical Physics*, **21**, 2124–2130.
Frenkel, A. (1980). *Physical Review D*, **21**, 2986–2987.
Gotay, M. (1983). *Journal of Physics A*, **16**, L143–L145.
Grácia, X., and Pons, J. M. (1988). *Annals of Physics*, **187**, 355–368.
Kobe, D. H. (1981). *American Journal of Physics*, **49**, 581–588.
Li, Z. P. (1981). *Acta Physica Sinica*, **30**, 1599–1671.
Li, Z. P. (1984). *Acta Physica Sinica*, **33**, 814–825.
Li, Z. P. (1987). *International Journal of Theoretical Physics*, **26**, 853–860.
Li, Z. P., and Li, X. (1990). *International Journal of Theoretical Physics*, **29**, 765–771.
Rosen, J. (1974*a*). *Annals of Physics*, **82**, 54–69.
Rosen, J. (1974*b*). *Annals of Physics*, **82**, 70–88.
Sugano, R. (1982). *Progress of Theoretical Physics*, **68**, 1377–1393.
Sugano, R., and Kamo, H. (1982). *Progress of Theoretical Physics*, **67**, 1966–1988.
Sugano, R., and Kimura, T. (1983*a*). *Progress of Theoretical Physics*, **69**, 252–261.
Sugano, R., and Kimura, T. (1983*b*). *Journal of Physics A*, **16**, 4417–4421.
Sundermeyer, K. (1982). *Constrained Dynamics*, Springer-Verlag, Berlin.